Journal of Statistical Physics, Vol. 54, Nos. 5/6, 1989

The Distribution of Exit Times for Weakly Colored Noise

Patrick S. Hagan,[1] Charles R. Doering,[2] and C. David Levermore[3]

Received June 14, 1988; revision received September 23, 1988

We analyze the exit time (first passage time) problem for the Ornstein–Uhlenbeck model of Brownian motion. Specifically, consider the position $X(t)$ of a particle whose velocity is an Ornstein–Uhlenbeck process with amplitude σ/ε and correlation time ε^2,

$$dX/dt = \sigma Z/\varepsilon, \qquad dZ/dt = -Z/\varepsilon^2 + 2^{1/2}\xi(t)/\varepsilon$$

where $\xi(t)$ is Gaussian white noise. Let the exit time t_{ex} be the first time the particle escapes an interval $-A < X(t) < B$, given that it starts at $X(0) = 0$ with $Z(0) = z_0$. Here we determine the exit time probability distribution $F(t) \equiv \text{Prob}\{t_{ex} > t\}$ by directly solving the Fokker–Planck equation. In brief, after taking a Laplace transform, we use singular perturbation methods to reduce the Fokker–Planck equation to a boundary layer problem. This boundary layer problem turns out to be a half-range expansion problem, which we solve via complex variable techniques. This yields the Laplace transform of $F(t)$ to within a transcendentally small $O(e^{-A/\varepsilon\sigma} + e^{-B/\varepsilon\sigma})$ error. We then obtain $F(t)$ by inverting the transform order by order in ε. In particular, by letting $B \to \infty$ we obtain the solution to Wang and Uhlenbeck's unsolved problem b; through $O(\varepsilon^2\sigma^2/A^2)$ this solution is

$$F(t) = \text{Erf}\left\{\frac{A + \varepsilon\sigma\alpha + \varepsilon\sigma z_0}{2\sigma(t - \varepsilon^2\kappa)^{1/2}}\right\} + \cdots \qquad \text{for } \frac{t}{\varepsilon^2} \gg 1$$

and $F = 1$ otherwise. Here, $\alpha = |\zeta(1/2)| = 1.4603...$, where ζ is the Riemann zeta function, and the constant κ is 0.22749... .

KEY WORDS: Exit times; first passage times; colored noise; Ornstein–Uhlenbeck process; half-range expansion; singular perturbation methods.

---

[1] Theoretical Division, MS B284, Los Alamos National Laboratory, Los Alamos, New Mexico 87545.

[2] Department of Physics, Clarkson University, Potsdam, New York 13676.

[3] Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

0022-4715/89/0300-1321$06.00/0

1. INTRODUCTION

Many random physical processes are naturally modeled as stationary, Markovian, and, in view of the central limit theorem, Gaussian.[1-4] By Doob's theorem,[5] the process is then an Ornstein–Uhlenbeck process. So consider the Ornstein–Uhlenbeck process $Z(t)$ defined by

$$dZ/dt = -Z/\varepsilon^2 + 2^{1/2}\xi(t)/\varepsilon \tag{1.1}$$

where $\xi(t)$ is unit-strength Gaussian white noise. The transition probability density is[4]

$$\text{Pd}\{Z(t) = z \mid Z(0) = z_0\} = \frac{1}{(2\pi\rho)^{1/2}} e^{-(z - mz_0)^2/2\rho} \tag{1.2a}$$

where

$$m(t) = e^{-t/\varepsilon^2}, \qquad \rho(t) = 1 - e^{-2t/\varepsilon^2} \tag{1.2b}$$

which yields the correlation time $t_{\text{cor}} = \varepsilon^2$.

Consider the position $X(t)$ of a particle whose velocity is an Ornstein–Uhlenbeck process,

$$dX/dt = \sigma Z/\varepsilon \tag{1.3a}$$

$$dZ/dt = -Z/\varepsilon^2 + 2^{1/2}\xi(t)/\varepsilon \tag{1.3b}$$

The non-Markovian position process $X(t)$ is the model of Brownian motion introduced by Ornstein and Uhlenbeck in 1930.[6] In the white-noise (high-friction) limit $\varepsilon^2 \to 0$, $X(t)$ is a Gaussian Markov process with diffusion coefficient σ^2. In this paper we analyze the exit time (first passage time) problem for $X(t)$. Specifically, let the exit time t_{ex} be the first time the particle escapes the interval $-A < X(t) < B$, given that it starts at $X(0) = 0$ with $Z(0) = z_0$. We shall determine the exit time probability distribution

$$F(t) = \text{Prob}\{t_{\text{ex}} > t\} \tag{1.4}$$

in the asymptotic limit of "small" correlation times ε^2.

To be more precise, the joint process (X, Z) *is* Markovian and its transition density satisfies the Fokker–Planck equation

$$\varepsilon^2 p_t + \varepsilon\sigma z p_x = p_{zz} + z p_z + p \tag{1.5}$$

[4] Throughout we use the notation Pd to refer to probability densities and Prob to refer to probabilities.

If the particle were free to wander over all of R^1, then we could solve (1.5) in the absence of any boundary conditions and obtain the free space solution

$$p^F(t, x, z) = \mathrm{Pd}\{X(t) = x, Z(t) = z \mid X(0) = 0, Z(0) = z_0\} \tag{1.6}$$

This yields

$$p^F = \frac{1}{(2\pi\rho)^{1/2}} e^{-(z - mz_0)^2/2\rho} \frac{1}{(2\pi\Omega)^{1/2}} e^{-[x - \varepsilon\sigma c(z + z_0)]^2/2\Omega} \tag{1.7a}$$

with $m(t)$ and $\rho(t)$ given in (1.2b), and

$$c(t) = \tanh(t/2\varepsilon^2), \qquad \Omega(t) = 2\sigma^2 t - 4\varepsilon^2\sigma^2 \tanh(t/2\varepsilon^2) \tag{1.7b}$$

When $t/\varepsilon^2 \gg 1$, this becomes

$$p^F = \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \frac{1}{[4\pi\sigma^2(t - 2\varepsilon^2)]^{1/2}} e^{-[x - \varepsilon\sigma(z + z_0)]^2/4\sigma^2(t - 2\varepsilon^2)} \tag{1.8}$$

to within a transcendentally small error. Clearly the key time scale is the correlation time (mean free time) ε^2, and the key length scale is the persistence length (mean free length) $\varepsilon\sigma$. Crudely speaking, if $Z(0) = z_0$, then the particle moves an average distance $\varepsilon\sigma z_0$ before its initial velocity is "forgotten." We assume that the length scales have been nondimensionalized so that A and B are both dimensionles and $O(1)$, and we shall obtain $F(t)$ using asymptotic methods based on $\varepsilon\sigma \ll 1$. Thus, our results are valid whenever both $\varepsilon\sigma/A \ll 1$ and $\varepsilon\sigma/B \ll 1$; i.e., whenever the persistence length $\varepsilon\sigma$ is much less than the distance to either boundary. Alternatively, we can interpret this restriction as $\varepsilon^2 \ll \min\{A^2/\sigma^2, B^2/\sigma^2\}$, indicating a correlation time much shorter than the time scale for diffusion to either boundary.

1.1. Approach

Let $p(t, x, z)$ be the probability density that the particle is at $X(t) = x$, $Z(t) = z$, *and* has not escaped by time t. That is,

$$p(t, x, z) = \mathrm{Pd}\{X(t) = x, Z(t) = z, \textit{and } -A < X(t') < B \text{ for all } t' < t\} \tag{1.9}$$

given that $X(0) = 0$ and $Z(0) = z_0$. Then

$$F(t) = \mathrm{Prob}\{t_{\mathrm{ex}} > t\} = \int_{-A}^{B} \int_{-\infty}^{\infty} p(t, x, z)\, dz\, dx \tag{1.10}$$

So we need only find p. Now p satisfies the Fokker–Planck equation with absorbing boundary conditions,

$$\varepsilon^2 p_t + \varepsilon\sigma z p_x = p_{zz} + z p_z + p \quad \text{for} \quad -A < x < B, \quad \text{all } z \tag{1.11a}$$

$$p(t, -A, z) = 0 \quad \text{for} \quad z \geqslant 0 \tag{1.11b}$$

$$p(t, B, z) = 0 \quad \text{for} \quad z \leqslant 0 \tag{1.11c}$$

The boundary conditions arise because the particle has never left the interval. Thus, there is no chance of finding it with a *positive* velocity at $x = -A$ or with a *negative* velocity at $x = B$. See Fig. 1. No boundary conditions are required for the other half of the boundaries since they represent the particle leaving the interval. (These boundary conditions in phase space were first written down by Wang and Uhlenbeck.[7]) The initial condition for (1.11) is

$$p(0, x, z) = \delta(x)\,\delta(z - z_0) \tag{1.12}$$

From the free space solution (1.7) we can expect $p(t, x, z)$ to exhibit an initial transient on the ε^2 time scale, during which the particle forgets its initial velocity and attains its steady-state Gaussian velocity distribution.

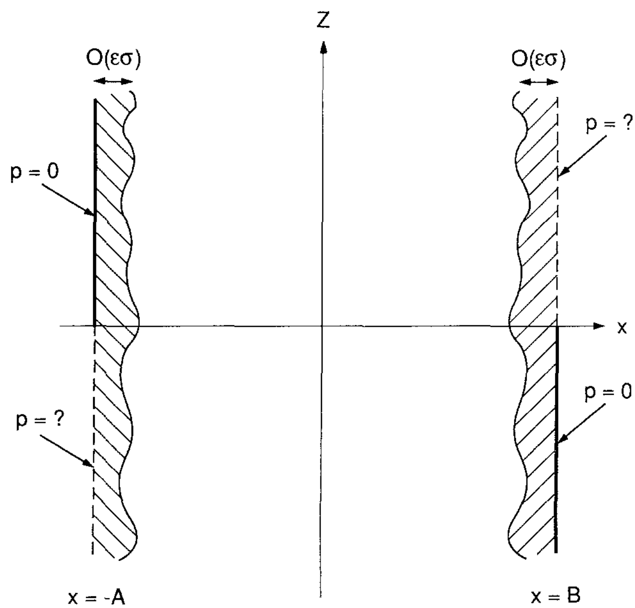

Fig. 1. The absorbing boundary conditions and boundary layers in velocity–position phase space.

Clearly, the behavior of p during this initial transient is irrelevant for calculating the exit time: even for times as large as $O(\varepsilon/\sigma)$ the probability of reaching either $x=-A$ or $x=B$ is only $e^{-O(1/\varepsilon\sigma)}$. See (1.8). Consequently, the boundaries are irrelevant for times $t=O(\varepsilon/\sigma)$ and smaller. So when $t=O(\varepsilon/\sigma)$, we have $p(t,x,z)=p^{\mathrm{F}}(t,x,z)$ with the free space solution $p^{\mathrm{F}}(t,x,z)$ given by (1.8), all to within an $e^{-O(1/\varepsilon\sigma)}$ error. Since (1.8) is the solution of the Fokker–Planck equation with the initial condition

$$p(t,x,z)=\frac{1}{(2\pi)^{1/2}}e^{-z^2/2}\delta(x-\varepsilon\sigma z-\varepsilon\sigma z_0) \qquad \text{at} \quad t=2\varepsilon^2 \tag{1.12'}$$

we can eliminate the initial transient by replacing the initial condition (1.12) with (1.12′). For all times longer than $O(\varepsilon/\sigma)$ this replacement leads only to a transcendentally small error in p, and thus $F(t)$. For all shorter times, $F(t)\equiv 1$ to within a transcendentally small error.

We will solve (1.11), (1.12′) by taking the Laplace transform

$$P(s,x,z)=\int_0^\infty e^{-st}p(t,x,z)\,dt \tag{1.13}$$

Then (1.11) and (1.12′) become

$$\varepsilon\sigma z P_x = P_{zz}+zP_z+(1-\varepsilon^2 s)P+\frac{\varepsilon^2}{(2\pi)^{1/2}}e^{-2\varepsilon^2 s}e^{-z^2/2}\delta(x-\varepsilon\sigma z-\varepsilon\sigma z_0)$$

$$\text{for} \quad -A<x<B, \quad \text{all } z \tag{1.14a}$$

$$P(s,-A,z)=0 \qquad \text{for} \quad z\geqslant 0 \tag{1.14b}$$

$$P(s,B,z)=0 \qquad \text{for} \quad z\leqslant 0 \tag{1.14c}$$

Physically, the velocity should be in its equilibrium Gaussian distribution everywhere except near the boundaries $x=-A$ and $x=B$. Near the boundaries, the velocity distribution must deviate from Gaussian in order to accommodate the absorbing boundary conditions. Thus, $P(s,x,z)$ should consist of an outer solution separating thin boundary layers of width $O(\varepsilon\sigma)$ next to the boundaries. See Fig. 1. After deriving the eigenfunctions for (1.14a) in Section 2, we find the outer solution to (1.14) in Section 3. Resolving the boundary layers then requires finding the solution to the Fokker–Planck equation which satisfies the absorbing boundary condition and properly matches the outer solution. This problem reduces to a half-range expansion problem for the Fokker–Planck operator, analogous to the Milne problem in classical transport theory.[8–11] We use the half-range expansion technique in ref. 12 to solve this problem in Appendix A, and then match the boundary layers to the outer solution in

Section 4. This yields $P(s, x, z)$, and thus the Laplace transform of $F(t)$, to within a transcendentally small $O(e^{-A/\varepsilon\sigma} + e^{-B/\varepsilon\sigma})$ error. Finally, we obtain the exit time distribution by inverting the transform order by order in ε in Section 5. Our results are summarized in the concluding Section 6, where we compare the colored-noise and white-noise exit time distributions.

1.2. Key Results

Here we discuss the reduced (marginal) probability density

$$p(t, x, *) \equiv \int_{-\infty}^{\infty} p(t, x, z)\, dz \tag{1.15}$$

and the exit time distribution $F(t)$.

When $t = O(\varepsilon/\sigma)$ the particle has only a transcendentally small chance of reaching either boundary, so $p(t, x, z)$ is given by (1.8). Hence,

$$p(t, x, *) = \frac{1}{[4\pi\sigma^2(t - 3\varepsilon^2/2)]^{1/2}}\, e^{-(x - \varepsilon\sigma z_0)^2/4\sigma^2(t - 3\varepsilon^2/2)} \qquad \text{for} \quad t = O\left(\frac{\varepsilon}{\sigma}\right) \tag{1.16}$$

For all larger times we obtain

$$p(t, x, *) = p^{\text{out}}(t, x, *) + p^A\left(t, \frac{A + x}{\varepsilon\sigma}, *\right) + p^B\left(t, \frac{B - x}{\varepsilon\sigma}, *\right) \tag{1.17}$$

where the terms p^A and p^B account for the boundary layers and are negligibly small unless x is within $O(\varepsilon\sigma)$ of the boundaries. Away from the boundaries we find that $p(t, x, *)$ is given by

$$p^{\text{out}}(t, x, *) = u(t - 3\varepsilon^2/2, x) + O(\varepsilon^3\sigma^3) \tag{1.18}$$

where u is the solution of the "effective" diffusion problem

$$u_t = \sigma^2 u_{xx} \qquad \text{for} \quad -A^* < x < B^* \tag{1.19a}$$

$$u = 0 \qquad \text{at} \quad x = -A^* \tag{1.19b}$$

$$u = 0 \qquad \text{at} \quad x = B^* \tag{1.19c}$$

$$u = \delta(x - \varepsilon\sigma z_0) \qquad \text{at} \quad t = 0 \tag{1.19d}$$

and the "apparent" boundary positions are

$$A^* = A + \varepsilon\sigma\alpha, \qquad B^* = B + \varepsilon\sigma\alpha \tag{1.20a}$$

with

$$\alpha = |\zeta(1/2)| = 1.4603545... \tag{1.20b}$$

Here ζ is the Riemann zeta function. As we shall see, the outward shift in the apparent boundary positions by the "Milne extrapolation length" $\varepsilon\sigma\alpha$[9-12] is caused directly by the boundary layers. See Fig. 2. Moreover, (1.16) shows that the $3\varepsilon^2/2$ time delay and the effective starting position $\varepsilon\sigma z_0$ are due to the initial transient.

There is only a transcendentally small chance of exiting before $t = O(\varepsilon/\sigma)$. For all later times we find that

$$F(t) = \int_{-A^*}^{B^*} u(t - \varepsilon^2\kappa, x)\, dx + \cdots \tag{1.21a}$$

where the constant κ is

$$\kappa = 0.2274981... \tag{1.21b}$$

$F(t)$ can now be obtained explicitly through $O(\varepsilon^2\sigma^2)$ by solving the diffusion problem (1.19). Explicit formulas are given in Section 5.

The reduction in the time delay from $3\varepsilon^2/2$ to $\varepsilon^2\kappa$ arises from two sources. For convenience, we are integrating over the extended interval $[-A^*, B^*]$ in (1.21a) instead of the true interval $[-A, B]$. Part of the

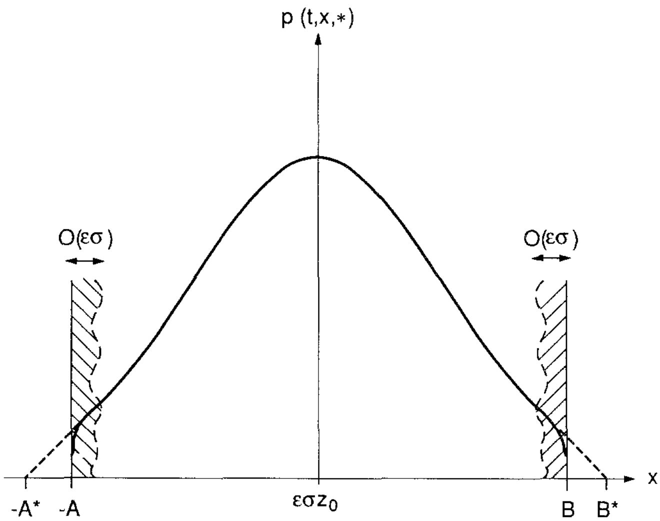

Fig. 2. Qualitative sketch of the reduced density. Shown are the boundary layers and the apparent boundary positions.

reduction represents the time needed to cross the extra distance $\varepsilon\sigma\alpha$, and compensates for the widened interval. The remainder arises because the velocity distribution is biased toward the boundary in the boundary layers (the particle cannot be coming *in* from the boundary), which reduces the exit time slightly.

We shall also find that the mean and variance of the exit time are

$$E\{t_{\text{ex}}\} = \frac{ab}{2\sigma^2} + \varepsilon^2 \kappa \tag{1.22a}$$

$$\text{Var}\{t_{\text{ex}}\} = \frac{ab}{12\sigma^4}(a^2+b^2) - \frac{\varepsilon^3}{6\sigma}\zeta(\tfrac{3}{2})(a+b) + \varepsilon^4\kappa' \tag{1.22b}$$

to within a transcendentally small error, where

$$a = A + \varepsilon\sigma\alpha + \varepsilon\sigma z_0, \qquad b = B + \varepsilon\sigma\alpha - \varepsilon\sigma z_0 \tag{1.22c}$$

are the distances from the effective starting position $\varepsilon\sigma z_0$ to the apparent boundaries, and

$$\kappa' = -0.2311372... \tag{1.22d}$$

In particular, (1.22a) is the mean first passage time formula discovered in ref. 12.

Finally, determining the exit time distribution for a semi-infinite region $-A < x < \infty$ is Wang and Uhlenbeck's unsolved problem b.[7] Its solution can be obtained by setting $B = +\infty$ and solving (1.19). This yields

$$F(t) = \text{Erf}\left\{\frac{A + \varepsilon\sigma\alpha + \varepsilon\sigma z_0}{2\sigma(t - \varepsilon^2\kappa)^{1/2}}\right\} + \cdots \tag{1.23}$$

through $O(\varepsilon^2\sigma^2)$. Note that this solution can also be deduced from ref. 11.

2. FORMULATION

We can simplify the ensuing calculations by redefining

$$t^{\text{new}} = \sigma^2 t^{\text{old}}, \qquad \varepsilon^{\text{new}} = \sigma\varepsilon^{\text{old}} \tag{2.1a}$$

and by defining $w(t, x, z)$ by

$$p(t, x, z) = \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} w(t, x, z) \tag{2.1b}$$

In terms of w and the new ε, (1.11) and (1.12′) now become

$$\varepsilon^2 w_t + \varepsilon z w_x = w_{zz} - z w_z \qquad \text{for} \quad -A < x < B, \quad \text{all } z \tag{2.2a}$$

$$w(t, -A, z) = 0 \qquad \text{for} \quad z \geqslant 0 \tag{2.2b}$$

$$w(t, B, z) = 0 \qquad \text{for} \quad z \leqslant 0 \tag{2.2c}$$

with

$$w(t, x, z) = \delta(x - \varepsilon z - \varepsilon z_0) \qquad \text{at} \quad t = 2\varepsilon^2 \tag{2.2d}$$

Taking the Laplace transform

$$W(s, x, z) = \int_0^\infty e^{-st} w(t, x, z)\, dt \tag{2.3}$$

now yields

$$\varepsilon z W_x = W_{zz} - z W_z - \varepsilon^2 s W + \varepsilon^2 e^{-2\varepsilon^2 s} \delta(x - \varepsilon z - \varepsilon z_0)$$
$$\text{for} \quad -A < x < B, \quad \text{all } z \tag{2.4a}$$

$$W(s, -A, z) = 0 \qquad \text{for} \quad z \geqslant 0 \tag{2.4b}$$

$$W(s, B, z) = 0 \qquad \text{for} \quad z \leqslant 0 \tag{2.4c}$$

It is clear from (2.4a) that the operator L,

$$Lv \equiv v_{zz} - z v_z - \varepsilon^2 s v \tag{2.5}$$

plays a central role. In the inner product

$$\langle u, v \rangle \equiv \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-z^2/2} u(z)\, v(z)\, dz \tag{2.6}$$

L is self-adjoint:

$$\langle u, Lv \rangle = \langle Lu, v \rangle \tag{2.7}$$

Moreover, the Laplace transform of the reduced density is now just

$$P(s, x, *) = \langle 1, W(s, x, z) \rangle \tag{2.8}$$

so the Laplace transform $\bar{F}(s)$ of the exit time distribution $F(t)$ is

$$\bar{F}(s) = \int_{-A}^{B} P(s, x, *)\, dx = \int_{-A}^{B} \langle 1, W(s, x, z) \rangle\, dx \tag{2.9}$$

We shall obtain the exit time distribution by solving (2.4) and then evaluating and inverting (2.9).

2.1. Eigenfunctions

To solve (2.4), we must consider the generalized eigenvalue problem

$$Lv \equiv v_{zz} - zv_z - \varepsilon^2 sv = \lambda zv \qquad \text{for} \quad -\infty < z < \infty \tag{2.10a}$$

$$\langle v, v \rangle < \infty \tag{2.10b}$$

We shall solve (2.4) by expanding in the eigenfunctions defined by (2.10).

The solution of (2.10a) with the appropriate asymptotics as $z \to +\infty$ is[13]

$$v(\lambda, z) = U(q, z + 2\lambda)\, e^{z^2/4} \qquad \text{with} \quad q = -1/2 - \lambda^2 + \varepsilon^2 s \tag{2.11}$$

where U is the parabolic cylinder function of index q. However, $v(\lambda, z) \sim e^{z^2/2}$ as $z \to -\infty$ unless $\lambda^2 = n + \varepsilon^2 s$ for a nonnegative integer n. So let

$$\lambda_n = (n + \varepsilon^2 s)^{1/2} \tag{2.12a}$$

Then the eigenvalues and corresponding eigenfunctions are

$$\lambda = +\lambda_n, \qquad v_n^+(z) \equiv U(z + 2\lambda_n)\, e^{z^2/4} = \mathrm{He}_n(z + 2\lambda_n)\, e^{-\lambda_n z - \lambda_n^2} \tag{2.12b}$$

$$\lambda = -\lambda_n, \qquad v_n^-(z) \equiv U(z - 2\lambda_n)\, e^{z^2/4} = \mathrm{He}_n(z - 2\lambda_n)\, e^{+\lambda_n z - \lambda_n^2} \tag{2.12c}$$

for $n = 0, 1, 2,...,$ where He_n are the Hermite polynomials. Here we are suppressing the index q of the parabolic cylinder functions; from now on it is to be understood as $-1/2 - \lambda^2 + \varepsilon^2 s$.

To determine orthogonality, let v_1 and v_2 be any two eigenfunctions of (2.10) with eigenvalues λ_1 and λ_2. Since L is self-adjoint,

$$(\lambda_1 - \lambda_2)\langle zv_1, v_2 \rangle = \langle Lv_1, v_2 \rangle - \langle v_1, Lv_2 \rangle = 0 \tag{2.13}$$

So if $\lambda_1 \neq \lambda_2$, then v_1 and v_2 satisfy the orthogonality relation

$$\langle zv_1, v_2 \rangle = 0 \tag{2.14}$$

Thus, we obtain[13]

$$\langle zv_n^-, v_m^- \rangle = 2\lambda_n n!\, \delta_{nm}, \qquad \langle zv_n^+, v_m^+ \rangle = -2\lambda_n n!\, \delta_{nm}, \qquad \langle zv_n^-, v_m^+ \rangle = 0 \tag{2.15}$$

Consequently, the expansion of a given function $f(z)$ is[14]

$$f(z) = \sum_{n=0}^{\infty} f_n^- v_n^-(z) - \sum_{n=0}^{\infty} f_n^+ v_n^+(z) \tag{2.16a}$$

with

$$f_n^- = \langle zv_n^-, f\rangle/2\lambda_n n!, \qquad f_n^+ = \langle zv_n^+, f\rangle/2\lambda_n n! \tag{2.16b}$$

For future reference we note the symmetry property

$$v_n^-(z) = (-1)^n v_n^+(-z) \tag{2.17}$$

We will also need

$$\langle 1, v_n^-\rangle = [-\lambda_n]^n e^{-\lambda_n^2/2}, \qquad \langle 1, v_n^+\rangle = [\lambda_n]^n e^{-\lambda_n^2/2} \tag{2.18}$$

as can be derived from Rodrigues' formula.[13]

3. THE OUTER SOLUTION

We now solve (2.4) to find the solution W^{out} which is valid away from the boundaries. A particular solution of (2.4) is

$$W^{\text{P}}(s, x, z) = \frac{1}{2\sqrt{s}} e^{-2\varepsilon^2 s} e^{-s^{1/2}|x - \varepsilon z - \varepsilon z_0|} \tag{3.1}$$

as can be verified by substitution. [This solution is just the Laplace transform of (1.8) in the new variables (2.1).] Thus, the general outer solution is

$$W^{\text{out}} = W^{\text{P}} + W^{\text{H}} \tag{3.2a}$$

where W^{H} satisfies the homogeneous equation

$$\varepsilon z W_x = W_{zz} - zW_z - \varepsilon^2 s W \equiv LW \tag{3.2b}$$

Since $\lambda_0 = \varepsilon\sqrt{s}$, expanding W^{H} in the eigenfunctions of L yields

$$W^{\text{H}} = \bar{C}(s)\, v_0^-(z)\, e^{-s^{1/2}x} + \bar{D}(s)\, v_0^+(z)\, e^{s^{1/2}x} + \sum_{n=1}^{\infty} C_n(s)\, v_n^-(z)\, e^{-\lambda_n x/\varepsilon} + \sum_{n=1}^{\infty} D_n(s)\, v_n^+(z)\, e^{+\lambda_n x/\varepsilon} \tag{3.3}$$

Clearly the coefficients C_n and D_n must be zero for all $n = 1, 2, \ldots$. For if W^{H} grew exponentially on the short x/ε scale as either $x/\varepsilon \to \infty$ or as $x/\varepsilon \to -\infty$, then the outer solution would become transcendentally large and could never match the boundary layer solutions. Rewriting

$$C(s) = -2s^{1/2} e^{\varepsilon^2 s - \varepsilon z_0 s^{1/2}} \bar{C}(s), \qquad D(s) = -2s^{1/2} e^{\varepsilon^2 s + \varepsilon z_0 s^{1/2}} \bar{D}(s) \tag{3.4}$$

for simplicity, the general solution to the outer problem now becomes

$$W^{\text{out}} = \frac{1}{2\sqrt{s}} e^{-2\varepsilon^2 s} \{ e^{-s^{1/2}|x - \varepsilon z - \varepsilon z_0|} - C e^{-s^{1/2}(x - \varepsilon z - \varepsilon z_0)} - D e^{+s^{1/2}(x - \varepsilon z - \varepsilon z_0)} \} \tag{3.5}$$

where the coefficients $C(s)$ and $D(s)$ must be found by matching to the boundary layers.

4. BOUNDARY LAYERS

Since W^{out} does not satisfy the boundary conditions (2.4b) and (2.4c), there must be boundary layers at $x = -A$ and $x = B$. To resolve the layer at $x = -A$, define

$$\bar{x} = (A + x)/\varepsilon \tag{4.1}$$

In terms of the boundary layer variable $\bar{x}$, (2.4) yields

$$zW_{\bar{x}} = W_{zz} - zW_z - \varepsilon^2 sW \qquad \text{for} \quad \bar{x} > 0, \quad \text{all } z \tag{4.2a}$$

$$W = 0 \qquad \text{at} \quad \bar{x} = 0, \quad \text{for } z \geqslant 0 \tag{4.2b}$$

and matching the outer solution (3.5) requires

$$W \sim \frac{1}{2\sqrt{s}} e^{-2\varepsilon^2 s} \{ (1 - D) e^{-s^{1/2}(A + \varepsilon z_0)} e^{\varepsilon s^{1/2}(\bar{x} - z)} - C e^{s^{1/2}(A + \varepsilon z_0)} e^{-\varepsilon s^{1/2}(\bar{x} - z)} \} \tag{4.2c}$$

for $\bar{x} \gg 1$.

To solve the boundary layer problem (4.2), suppose that we could solve the problem

$$zV_{\bar{x}} = V_{zz} - zV_z - \varepsilon^2 sV \qquad \text{for} \quad \bar{x} > 0, \quad \text{all } z \tag{4.3a}$$

$$V(s, 0, z) = e^{-\varepsilon s^{1/2} z} \qquad \text{for} \quad z \geqslant 0 \tag{4.3b}$$

using only the eigenfunctions $v_n^-(z)$:

$$V(s, \bar{x}, z) = \beta_0(s) e^{-\varepsilon s^{1/2}(\bar{x} - z)} + \sum_{n=1}^{\infty} \beta_n(s) v_n^-(z) e^{-\lambda_n \bar{x}} \tag{4.3c}$$

[In the first term of (4.3c), note that $e^{\varepsilon s^{1/2} z}$ is essentially just $v_0^-(z)$. See (2.12).] In terms of V, the solution of (4.2) must be

$$W = \frac{1}{2\sqrt{s}} e^{-2\varepsilon^2 s} (1 - D) e^{-s^{1/2}(A + \varepsilon z_0)} \{ e^{\varepsilon s^{1/2}(\bar{x} - z)} - V(s, \bar{x}, z) \} \tag{4.4}$$

and we must have

$$C = (1 - D)\, \beta_0(s)\, e^{-2s^{1/2}(A + \varepsilon z_0)} \tag{4.5}$$

for then (4.4) satisfies (4.2a) and (4.2b) exactly, and it reduces to the outer solution (4.2c) when $\bar{x} \gg 1$. Moreover, we can write (4.4) in the original space variable x as

$$W = W^{\text{out}}(s, x, z) + W^A\left(s, \frac{A + x}{\varepsilon}, z\right) \tag{4.6a}$$

where the outer solution is given by (3.5) and the contribution from the boundary layer is

$$W^A(s, \bar{x}, z) = \frac{1}{2\sqrt{s}}\, e^{-2\varepsilon^2 s}(1 - D)\, e^{-s^{1/2}(A + \varepsilon z_0)} \times \{\beta_0(s)\, e^{-\varepsilon s^{1/2}(\bar{x} - z)} - V(s, \bar{x}, z)\} \tag{4.6b}$$

Note that W^A is transcendentally small unless $\bar{x} \equiv (A + x)/\varepsilon$ is $O(1)$. Thus, if we can solve (4.3) for V, then we can resolve the boundary layer at $x = -A$ and obtain one relation, (4.5), for the unknown coefficients C and D.

The boundary layer at $x = B$ can be resolved similarly. This yields

$$W = W^{\text{out}}(s, x, z) + W^A\left(s, \frac{A + x}{\varepsilon}, z\right) + W^B\left(s, \frac{B - x}{\varepsilon}, z\right) \tag{4.7a}$$

where the contribution from the boundary layer at $x = B$ is

$$W^B(s, \bar{x}, z) = \frac{1}{2\sqrt{s}}\, e^{-2\varepsilon^2 s}(1 - C)\, e^{-s^{1/2}(B - \varepsilon z_0)} \times \{\beta_0(s)\, e^{-\varepsilon s^{1/2}(\bar{x} + z)} - V(s, \bar{x}, -z)\} \tag{4.7b}$$

and V is the same function as before. Resolving this boundary layer also yields

$$D = (1 - C)\, \beta_0(s)\, e^{-2s^{1/2}(B - \varepsilon z_0)} \tag{4.8}$$

So if we can solve problem (4.3), then (4.5) and (4.8) will determine $C(s)$ and $D(s)$, and (4.7a) will yield the uniformly valid solution to (2.4).

4.1. The Uniformly Valid Solution

In Appendix A we use the half-range expansion technique in ref. 12 to obtain the unique solution to (4.3). We discover that

$$\beta_0(s) = M^2(\varepsilon\sqrt{s}), \qquad \beta_n(s) = (-1)^n \varepsilon\sqrt{s}\, e^{\varepsilon^2 s} M(\varepsilon\sqrt{s})\, N(\lambda_n, \varepsilon^2 s)/\lambda_n n! \tag{4.9a}$$

where $\lambda_n = (n + \varepsilon^2 s)^{1/2}$ as before. Analogous to the Γ function, the function N is defined by an infinite product

$$N(\lambda, r^2) \equiv \prod_{k=1}^{\infty} [(1 + r^2/k)^{1/2} + \lambda/k^{1/2}]\, e^{-2\lambda[k^{1/2} - (k-1)^{1/2}]}\{(k+1)/k\}^{(\lambda^2 - r^2)/2} \tag{4.9b}$$

and

$$M(\varepsilon\sqrt{s}) \equiv N(\varepsilon\sqrt{s}, \varepsilon^2 s) \tag{4.9c}$$

The functions N and M are briefly examined in Appendix B. In particular, we find that

$$M(\varepsilon\sqrt{s}) = \exp\left\{\varepsilon\sqrt{s}\int_0^1 \zeta(\tfrac{1}{2}, 1 + \varepsilon^2 s q^2)\, dq\right\} \tag{4.10a}$$

and

$$N(\lambda_n, \varepsilon^2 s) = N(\sqrt{n}, 0) \exp\left\{\tfrac{1}{2}\varepsilon^2 s \int_0^1 \zeta(\tfrac{1}{2}, 1 + \varepsilon^2 s q)/(n + \varepsilon^2 s q)^{1/2}\, dq\right\} \tag{4.10b}$$

where ζ is the generalized Riemann zeta function.[15] Expanding now yields

$$M(\varepsilon\sqrt{s}) = e^{-\varepsilon\alpha s^{1/2}}\{1 - \tfrac{1}{6}\varepsilon^3 s^{3/2}\zeta(\tfrac{3}{2}) + \cdots\} \tag{4.11a}$$

$$N(\lambda_n, \varepsilon^2 s) = N(\sqrt{n}, 0)\, e^{-\varepsilon^2 s\alpha/2n^{1/2}}\{1 - \cdots\} \tag{4.11b}$$

where

$$\alpha = |\zeta(\tfrac{1}{2})| = 1.4603545... \tag{4.11c}$$

By solving (4.5) and (4.8) for C and D, we can now write the uniformly valid solution to (2.4). Let

$$\begin{aligned} Q(s) &\equiv M^2(\varepsilon\sqrt{s})\, e^{-2s^{1/2}(A + \varepsilon z_0)} \\ &= e^{-2s^{1/2}(A + \varepsilon\alpha + \varepsilon z_0)}\{1 - \tfrac{1}{3}\varepsilon^3 s^{3/2}\zeta(\tfrac{3}{2}) + \cdots\} \end{aligned} \tag{4.12a}$$

$$\begin{aligned} R(s) &\equiv M^2(\varepsilon\sqrt{s})\, e^{-2s^{1/2}(B - \varepsilon z_0)} \\ &= e^{-2s^{1/2}(B + \varepsilon\alpha - \varepsilon z_0)}\{1 - \tfrac{1}{3}\varepsilon^3 s^{3/2}\zeta(\tfrac{3}{2}) + \cdots\} \end{aligned} \tag{4.12b}$$

Then the uniformly valid solution is

$$W = W^{\text{out}}(s, x, z) + W^{A}\left(s, \frac{A+x}{\varepsilon}, z\right) + W^{B}\left(s, \frac{B-x}{\varepsilon}, z\right) \quad (4.13a)$$

where the outer solution is

$$W^{\text{out}} = \frac{1}{2\sqrt{s}} e^{-2\varepsilon^2 s} \left\{ e^{-s^{1/2}|x - \varepsilon z - \varepsilon z_0|} - \frac{1-R}{1-QR} Q e^{-s^{1/2}(x - \varepsilon z - \varepsilon z_0)} - \frac{1-Q}{1-QR} R e^{s^{1/2}(x - \varepsilon z - \varepsilon z_0)} \right\} \quad (4.13b)$$

and the modifications due to the boundary layers are

$$W^{A} = -\varepsilon e^{-\varepsilon^2 s} \frac{1-R}{1-QR} Q^{1/2} \sum_{n=1}^{\infty} (-1)^n \gamma_n v_n^-(z)\, e^{-\lambda_n(A+x)/\varepsilon} \quad (4.13c)$$

$$W^{B} = -\varepsilon e^{-\varepsilon^2 s} \frac{1-Q}{1-QR} R^{1/2} \sum_{n=1}^{\infty} (-1)^n \gamma_n v_n^-(-z)\, e^{-\lambda_n(B-x)/\varepsilon} \quad (4.13d)$$

Here,

$$\gamma_n = N(\lambda_n, \varepsilon^2 s)/2\lambda_n n! \quad (4.13e)$$

Clearly, W satisfies Eq. (2.4a) exactly, since each term does. At the left boundary, however, W^A cancels out only the outer solution. Thus,

$$W(s, -A, z) = W^{B}\left(s, \frac{A+B}{\varepsilon}, z\right) = O(e^{-(A+B)/\varepsilon}) \qquad \text{for} \quad z \geqslant 0 \quad (4.14a)$$

Similarly,

$$W(s, B, z) = W^{A}\left(s, \frac{A+B}{\varepsilon}, z\right) = O(e^{-(A+B)/\varepsilon}) \qquad \text{for} \quad z \leqslant 0 \quad (4.14b)$$

So although (4.13) is not exact, it is valid to within a transcendentally small error.

5. THE REDUCED DENSITY. EXIT TIMES

Our remaining task is to integrate (4.13) to obtain the reduced density and the exit time distribution. Recall from (2.8) that the Laplace transform of the reduced density is

$$P(s, x, *) = \langle 1, W(s, x, z) \rangle \quad (5.1)$$

Clearly, the reduced density has the same boundary layer structure as W,

$$P(s, x, *) = P^{\text{out}}(s, x, *) + P^{A}\left(s, \frac{A+x}{\varepsilon}, *\right) + P^{B}\left(s, \frac{B-x}{\varepsilon}, *\right) \quad (5.2)$$

Obtaining $P^{\text{out}}(s, x, *)$ requires evaluating $\langle 1, \Phi(s, x, z)\rangle$, where

$$\Phi(s, x, z) = \frac{1}{2\sqrt{s}}\, e^{-2\varepsilon^2 s} e^{-s^{1/2}\,|x-\varepsilon z-\varepsilon z_0|} \quad (5.3a)$$

See (4.13b). This can readily be done by taking the inverse Laplace transform of (5.3a), evaluating the inner product in the time domain, and then taking its Laplace transform. This shows that we may set

$$\langle 1, \Phi(s, x, z)\rangle = \frac{1}{2\sqrt{s}}\, e^{-3\varepsilon^2 s/2} e^{-s^{1/2}\,|x-\varepsilon z_0|} \quad (5.3b)$$

Thus,

$$P^{\text{out}} = \frac{1}{2\sqrt{s}}\, e^{-3\varepsilon^2 s/2}\left\{ e^{-s^{1/2}\,|x-\varepsilon z_0|} - \frac{1-R}{1-QR}\, Q e^{-s^{1/2}(x-\varepsilon z_0)} - \frac{1-Q}{1-QR}\, R e^{s^{1/2}(x-\varepsilon z_0)} \right\} \quad (5.4a)$$

Moreover, using (2.18) shows that the boundary layer terms are

$$P^{A} = -\varepsilon e^{-3\varepsilon^2 s/2}\, \frac{1-R}{1-QR}\, Q^{1/2} \sum_{n=1}^{\infty} \gamma_n \lambda_n^n e^{-n/2} e^{-\lambda_n (A+x)/\varepsilon} \quad (5.4b)$$

$$P^{B} = -\varepsilon e^{-3\varepsilon^2 s/2}\, \frac{1-Q}{1-QR}\, R^{1/2} \sum_{n=1}^{\infty} \gamma_n \lambda_n^n e^{-n/2} e^{-\lambda_n (B-x)/\varepsilon} \quad (5.4c)$$

Now, the position process $X(t)$ is often modeled as a *Markovian* diffusion process, which amounts to approximating the reduced density $p(t, x, *)$ by an "effective" diffusion equation.[16–19] Such an approach cannot be expected to handle the boundary layers. So let us find the diffusion problem which correctly yields the outer solution (5.4a), and thus gives the correct reduced density away from the boundaries. Observe that $P^{\text{out}}(s, x, *)$ satisfies

$$P_{xx} - sP = -\delta(x - \varepsilon z_0)\, e^{-3\varepsilon^2 s/2} \quad (5.5)$$

Additionally, $P^{\text{out}}(s, x, *)$ is very nearly zero at the "extrapolated" boundary positions $x = -A^*$ and $x = B^*$, where

$$A^* \equiv A + \varepsilon\alpha, \qquad B^* \equiv B + \varepsilon\alpha \quad (5.6)$$

Specifically, (4.12) shows that P^{out} satisfies

$$P = \tfrac{1}{6}\varepsilon^3\zeta(\tfrac{3}{2})\, P_{xxx} - \cdots \qquad \text{at} \quad x = -A^* \tag{5.7a}$$

$$P = -\tfrac{1}{6}\varepsilon^3\zeta(\tfrac{3}{2})\, P_{xxx} + \cdots \qquad \text{at} \quad x = B^* \tag{5.7b}$$

Therefore, away from the boundaries the reduced density is

$$p^{\text{out}}(t, x, *) = u(t - \tfrac{3}{2}\varepsilon^2, x) \tag{5.8}$$

where u is the solution to the "effective" diffusion problem

$$u_t = u_{xx} \qquad \text{for} \quad -A^* < x < B^* \tag{5.9a}$$

$$u = \tfrac{1}{6}\varepsilon^3\zeta(\tfrac{3}{2})\, u_{xxx} - \cdots \qquad \text{at} \quad x = -A^* \tag{5.9b}$$

$$u = -\tfrac{1}{6}\varepsilon^3\zeta(\tfrac{3}{2})\, u_{xxx} + \cdots \qquad \text{at} \quad x = B^* \tag{5.9c}$$

with the initial condition

$$u(0, x) = \delta(x - \varepsilon z_0) \tag{5.9d}$$

Clearly the main effect of the boundary layers is to shift the "apparent" boundary positions outward by the "Milne extrapolation length" $\varepsilon\alpha$.

Recall that we rescaled both the time variable t and ε to simplify the calculations. See (2.1a). In the original t and ε, (5.8) and (5.9) agree with the results quoted in the introduction. For example, the diffusion coefficient in (5.9a) is σ^2 and the Milne extrapolation length is $\varepsilon\sigma\alpha$ in the original variables.

5.1. Exit Times

In this section we revert to the original variables t and ε. The exit time distribution derived here corresponds directly to the problem as originally posed in Section 1.

From Section 1.1 recall that *after the initial transient dies away*, say, when $t = O(\varepsilon/\sigma)$ or larger, the exit time distribution can be obtained to within a transcendentally small error by inverting

$$\bar{F}(s) = \int_{-A}^{B} P(s, x, *)\, dx \tag{5.10}$$

Integrating (5.2) and (5.4) then yields

$$\bar{F}(s) = \frac{1}{s} e^{-3\varepsilon^2/2} - \frac{1}{s} e^{-3\varepsilon^2 s/2} \frac{Q^{1/2} + R^{1/2}}{1 + (QR)^{1/2}} \times \{\tfrac{1}{2}[M(\varepsilon\sqrt{s})^{-1} + M(\varepsilon\sqrt{s})] + \varepsilon^2 s A(\varepsilon^2 s)\} \tag{5.11a}$$

in the original variables, where Q and R are now

$$Q = M^2(\varepsilon\sqrt{s})\, e^{-2s^{1/2}(A+\varepsilon\sigma z_0)/\sigma} = e^{-2as^{1/2}/\sigma}\{1 - \tfrac{1}{3}\varepsilon^3 s^{3/2}\zeta(\tfrac{3}{2}) + \cdots\} \quad (5.11b)$$

$$R = M^2(\varepsilon\sqrt{s})\, e^{-2s^{1/2}(B-\varepsilon\sigma z_0)/\sigma} = e^{-2bs^{1/2}/\sigma}\{1 - \tfrac{1}{3}\varepsilon^3 s^{3/2}\zeta(\tfrac{3}{2}) + \cdots\} \quad (5.11c)$$

Here

$$\Lambda(\varepsilon^2 s) \equiv \sum_{n=1}^{\infty} N(\lambda_n, \varepsilon^2 s)\, \lambda_n^{n-2} e^{-n/2}/2n! \quad (5.11d)$$

accounts for the modification in the exit time due to the boundary layers, and

$$a \equiv A + \varepsilon\sigma\alpha + \varepsilon\sigma z_0, \qquad b \equiv B + \varepsilon\sigma\alpha - \varepsilon\sigma z_0 \quad (5.12)$$

are the distances from the effective starting point $\varepsilon\sigma z_0$ to the apparent boundaries.

Moreover, there is only a transcendentally small chance of exiting, and thus $F(t) \equiv 1$ to within a transcendentally small error, when $t = O(\varepsilon/\sigma)$ or smaller. Combining this with (5.11), the exit time distribution $F(t)$ can be found to within a transcendentally small error for *all* times by inverting

$$\bar{F}(s) = \frac{1}{s} - \frac{1}{s} e^{-3\varepsilon^2 s/2} \frac{Q^{1/2} + R^{1/2}}{1 + (QR)^{1/2}} \times \{\tfrac{1}{2}[M(\varepsilon\sqrt{s})^{-1} + M(\varepsilon\sqrt{s})] + \varepsilon^2 s\Lambda(\varepsilon^2 s)\} \quad (5.13)$$

We have been unable to invert (5.13) in its entirety. We can expand (5.13) to any order in ε, and then invert to obtain $F(t)$ to that accuracy. Here we choose to expand through $O(\varepsilon^2)$. Then

$$\bar{F}(s) = \frac{1}{s} - \frac{1}{s}\frac{\cosh s^{1/2}(b-a)/2\sigma}{\cosh s^{1/2}(b+a)/2\sigma}\{1 - \varepsilon^2 s\kappa + \cdots\} \quad (5.14a)$$

with

$$\kappa = \frac{3}{2} - \frac{1}{2}\alpha^2 - \Lambda(0) = 0.2274981... \quad (5.14b)$$

Inverting (5.14) now gives

$$F(t) = H\left(\frac{\sigma^2(t - \varepsilon^2\kappa)}{(a+b)^2}\right) + \cdots \qquad \text{for } \frac{t}{\varepsilon^2} \gg 1 \quad (5.15a)$$

where

$$H(\theta)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left\{(2k+1)\frac{\pi a}{a+b}\right\}e^{-\pi^2(2k+1)^2\theta} \quad (5.15b)$$

Note that when θ is larger than, say, 1/10, the first term in (5.15b) suffices. See below, Fig. 4.

Instead of expanding (5.13) in powers of ε, we can expand in powers of s. Since

$$\bar{F}(s)=\int_0^\infty e^{-st}\,\mathrm{Prob}\{t_{ex}>t\}\,dt=\sum_{k=0}^{\infty}(-1)^k E\{t_{ex}^{k+1}\}\,s^k/(k+1)! \quad (5.16)$$

this yields the moments of t_{ex}. For example, the first two moments are

$$E\{t_{ex}\}=\frac{ab}{2\sigma^2}+\varepsilon^2\kappa \quad (5.17a)$$

and

$$\mathrm{Var}\{t_{ex}\}=\frac{ab}{12\sigma^4}(a^2+b^2)-\frac{\varepsilon^3}{6\sigma}\zeta(\tfrac{3}{2})(a+b)+\varepsilon^4\kappa' \quad (5.17b)$$

to within a transcendentally small error, where the constant κ' is

$$\kappa'=\tfrac{1}{3}\alpha\zeta(\tfrac{3}{2})-\tfrac{1}{6}\alpha^4-\alpha^2 A(0)+2A'(0)-A^2(0)=-0.2311372... \quad (5.17c)$$

5.2. Half-Space Exit Times

Finally, finding the exit time distribution for a semi-infinite interval $-A<x<+\infty$ is Wang and Uhlenbeck's problem b.[7] Setting $B=+\infty$ in (5.13) yields

$$\bar{F}(s)=\frac{1}{s}-\frac{1}{s}e^{-3\varepsilon^2 s/2}Q^{1/2}\{\tfrac{1}{2}[M(\varepsilon\sqrt{s})^{-1}+M(\varepsilon\sqrt{s})]+\varepsilon^2 sA(\varepsilon^2 s)\} \quad (5.18)$$

Expanding (5.18) and inverting the transform, we now find that the solution to problem b is

$$F(t)=G+\tfrac{1}{6}\varepsilon^3\sigma^3\zeta(\tfrac{3}{2})\frac{d^3G}{da^3}+\tfrac{1}{2}\varepsilon^4\sigma^4\kappa'\frac{d^4G}{da^4}+\cdots \qquad \text{for } \frac{t}{\varepsilon^2}\gg 1 \quad (5.19a)$$

where

$$G\equiv \mathrm{Erf}(a/2\sigma(t-\varepsilon^2\kappa)^{1/2}) \quad (5.19b)$$

6. CONCLUSIONS

The exit time distribution for a particle driven by weakly-colored Gaussian noise is given in Section 5. It is instructive to compare this distribution with the results for the "white-noise" case ($t_{\text{cor}} \equiv \varepsilon^2 = 0$) in order to see the consequences of a nonzero correlation time. For simplicity we consider only the exit time from the interval $[-L, L]$, assuming that the particle starts at the center, $X(0) = 0$, with velocity $Z(0) = z_0 = 0$.

For a particle driven by white noise,

$$dX/dt = 2^{1/2}\sigma\xi(t) \tag{6.1}$$

we need to solve the Fokker–Planck equation with absorbing boundary conditions:

$$\rho_t = \sigma^2 \rho_{xx} \qquad \text{for} \quad |x| < L \tag{6.2a}$$

$$\rho = 0 \qquad \text{at} \quad x = \pm L \tag{6.2b}$$

$$\rho(0, x) = \delta(x) \qquad \text{at} \quad t = 0 \tag{6.2c}$$

Then the exit time distribution and mean exit time are given by

$$F_{\text{w}}(t) = \text{Prob}\{t_{\text{ex}} > t\} = \int_{-L}^{L} \rho(t, x)\, dx \qquad \text{(white noise)} \tag{6.3a}$$

$$T_{\text{white}} \equiv E\{t_{\text{ex}}\} = \int_0^\infty \int_{-L}^{L} \rho(t, x)\, dx\, dt \qquad \text{(white noise)} \tag{6.3b}$$

Solving (6.2) yields the mean exit time $T_{\text{white}} = L^2/2\sigma^2$. This is also the white-noise diffusion time scale, so it is natural to nondimensionalize the time variable t in terms of T_{white},

$$\tau = t/T_{\text{white}} = 2\sigma^2 t/L^2 \tag{6.4}$$

In terms of τ the exit time distribution is then

$$F_{\text{w}}(\tau) = \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} e^{-(2n+1)^2 \pi^2 \tau/8} \qquad \text{(white noise)} \tag{6.5a}$$

Consequently, the exit time probability density is

$$f_{\text{w}}(\tau) \equiv -\frac{d}{d\tau} F_{\text{w}}(\tau) = \frac{\pi}{2} \sum_{n=0}^{\infty} (-1)^n (2n+1)\, e^{-(2n+1)^2 \pi^2 \tau/8} \qquad \text{(white noise)} \tag{6.5b}$$

For a particle driven by colored noise, the mean exit time is

$$T \equiv E\{t_{ex}\} = (L + \varepsilon\sigma\alpha)^2/2\sigma^2 + \varepsilon^2\kappa \qquad \text{(colored noise)} \tag{6.6}$$

to within a transcendentally small error, where $\alpha = |\zeta(1/2)| = 1.4603...$ and $\kappa = 0.22749...$. See (5.17). In terms of τ and the dimensionless correlation time

$$\eta = t_{cor}/T_{white} = 2\sigma^2\varepsilon^2/L^2 \tag{6.7}$$

this is

$$T/T_{white} \equiv E\{\tau_{ex}\} = [1 + (\eta\alpha^2/2)^{1/2}]^2 + \eta\kappa \qquad \text{(colored noise)} \tag{6.8a}$$

Moreover, through $O(\varepsilon^2\sigma^2/L^2)$ the exit time distribution $F(\tau)$ is

$$F(\tau) = \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} e^{-(2n+1)^2 \pi^2 (\tau - \eta\kappa)/8\Delta} \qquad \text{(colored noise)} \tag{6.8b}$$

where

$$\Delta = (1 + \varepsilon\sigma\alpha/L)^2 = [1 + (\eta\alpha^2/2)^{1/2}]^2 \tag{6.8c}$$

Consequently, the exit time probability density is

$$f(\tau) \equiv -\frac{d}{d\tau} F(\tau) = \frac{\pi}{2\Delta} \sum_{n=0}^{\infty} (-1)^n (2n+1)\, e^{-(2n+1)^2 \pi^2 (\tau - \eta\kappa)/8\Delta} \qquad \text{(colored noise)} \tag{6.8d}$$

Of course, (6.8) holds only for times $t/\varepsilon^2 \equiv \tau/\eta \gg 1$; at all earlier times the probability f of exiting is zero to within a transcendentally small error.

The finite-correlation-time results (6.8) are simply the white-noise results (6.5) with the dimensionless time τ delayed by $\eta\kappa$ and rescaled by Δ. Since Δ depends on the square root of η, the exit time depends very sensitively on the correlation time. Even small values of η can cause surprisingly large increases in the exit time as compared to the white noise limit. This is exhibited in Fig. 3, where the mean exit time T is graphed against η. Note that T is already 50% larger than T_{white} when the correlation time is only 5% of T_{white} ($\eta = 0.05$).

Figures 4a–4c show the effect of increasing the correlation time on the exit time probability density. There we plot the probability density $f(\tau)$ for the cases (a) $\eta = 0$ (white noise), (b) $\eta = 0.02$, and (c) $\eta = 0.10$. Even for "nearly" white noise, where t_{cor} is only 2% of T_{white} (case b), there is a

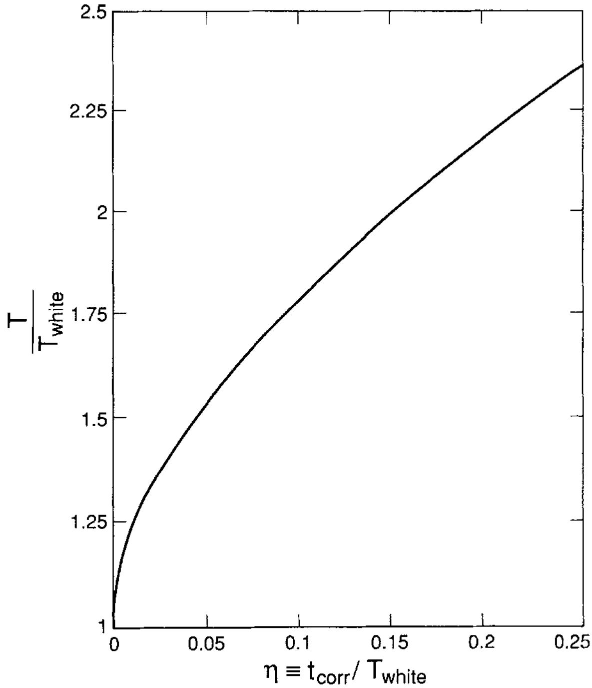

Fig. 3. The mean exit time T, normalized by the white noise value T_{white}, graphed against the dimensionless correlation time $\eta = t_{\text{cor}}/T_{\text{white}} = 2\varepsilon^2\sigma^2/L^2$.

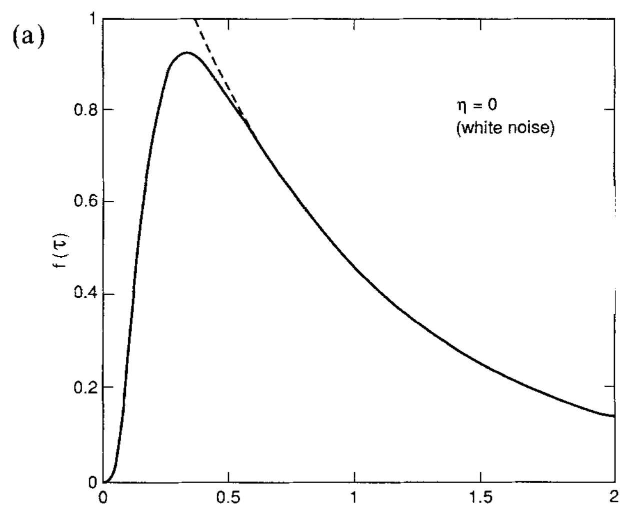

Fig. 4. The exit time probability density $f(\tau)$ versus the dimensionless time τ. Shown are the probability densities for (a) $\eta = 0$ (white noise), (b) $\eta = 0.02$, and (c) $\eta = 0.10$.

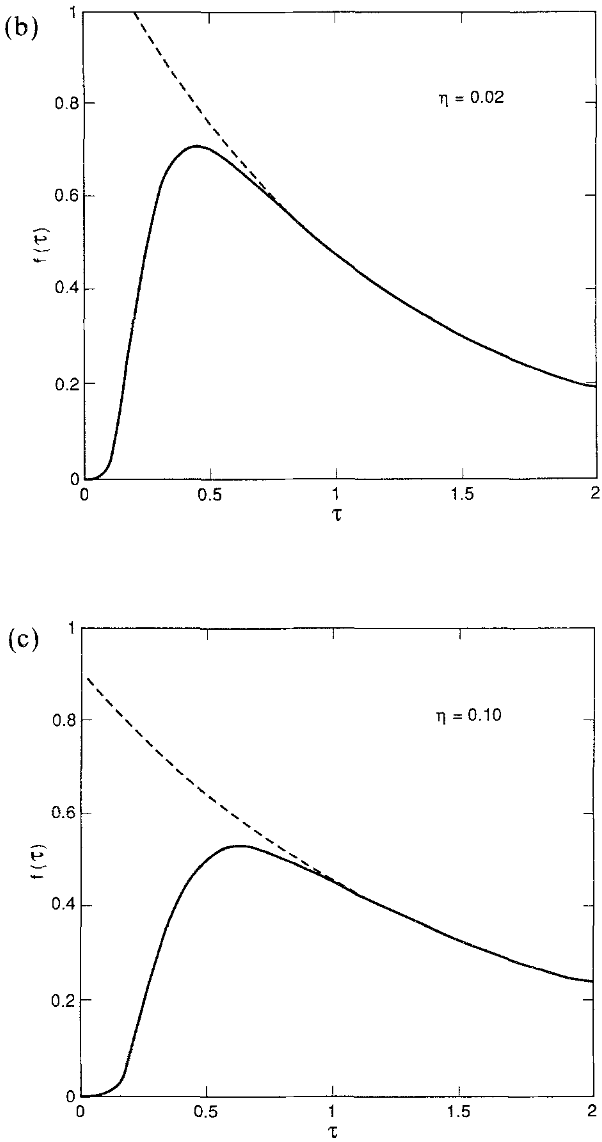

Fig. 4 (*continued*)

striking change in the distribution from its white-noise limit. As the correlation time increases further, the main effect is to broaden the peak in the density by the factor Δ. Finally, note that when τ/Δ is larger than 3/4, $f(\tau)$ can be accurately approximated by using only the first term in (6.8d). The exponential approximation obtained by using just this first term is shown by the dashed lines in Figs. 4a–4c.

APPENDIX A. THE HALF-RANGE EXPANSION

Here we solve the key boundary layer problem

$$zV_{\bar{x}} = V_{zz} - zV_z - \varepsilon^2 sV \qquad \text{for} \quad \bar{x} > 0, \quad \text{all } z \tag{A.1a}$$

$$V(s, 0, z) = e^{-\varepsilon s^{1/2} z} \qquad \text{for} \quad z \geqslant 0 \tag{A.1b}$$

with V of the form

$$V(s, \bar{x}, z) = \beta_0 e^{-\varepsilon s^{1/2}(\bar{x} - z)} + \sum_{n=1}^{\infty} \beta_n v_n^-(z)\, e^{-\lambda_n \bar{x}} \qquad \text{for all } z \tag{A.1c}$$

Note that at $\bar{x} = 0$,

$$V(s, 0, z) = \beta_0 e^{+\varepsilon s^{1/2} z} + \sum_{n=1}^{\infty} \beta_n v_n^-(z) = \begin{cases} e^{-\varepsilon s^{1/2} z} & \text{for} \quad z \geqslant 0 \\ \text{unknown} & \text{for} \quad z \leqslant 0 \end{cases} \tag{A.2}$$

Thus, (A.1) is a half-range expansion problem: We must match a prescribed function over half the domain using only half the eigenfunctions.

We shall solve (A.1) using the Laplace transform technique in ref. 12. Consider the Laplace transform with respect to $\bar{x}$,

$$H(s, \lambda, z) \equiv \int_0^{\infty} e^{-\lambda \bar{x}} V(s, \bar{x}, z)\, d\bar{x} \tag{A.3}$$

Transforming (A.1c) yields

$$H = \frac{\beta_0}{\lambda + \varepsilon\sqrt{s}}\, e^{+\varepsilon s^{1/2} z} + \sum_{n=1}^{\infty} \frac{\beta_n}{\lambda + \lambda_n}\, v_n^-(z) \qquad \text{for all } z \tag{A.4}$$

where the coefficients β are unknown. Alternatively, when $z \geqslant 0$ we can transform (A.1a) and (A.1b) and find

$$H_{zz} - zH_z - (\lambda z + \varepsilon^2 s) H = -z e^{-\varepsilon s^{1/2} z} \qquad \text{for all } z \geqslant 0 \tag{A.5}$$

Solving (A.5) yields

$$H = \frac{e^{-\varepsilon s^{1/2} z}}{\lambda - \varepsilon\sqrt{s}} - F(\lambda, s)\, U(q, z + 2\lambda)\, e^{z^2/4} \qquad \text{for} \quad z \geqslant 0 \tag{A.6a}$$

where the function $F(\lambda, s)$ is unknown and U is the parabolic cylinder function[13] of index

$$q = -\lambda^2 + \varepsilon^2 s - 1/2 \tag{A.6b}$$

Note that U is an entire function of λ.

We shall solve (A.1) by inspection: Since (A.6a) and (A.4) represent the same function, (A.6) can have no singularities in λ not present in (A.4). Thus, comparing (A.6a) with (A.4) will identify the singularities of F. Factoring out these singularities will then leave us with an entire function of λ. Comparing the asymptotics of (A.4) and (A.6a) as $|\lambda| \to \infty$ will then identify this entire function.

Since (A.4) does not have a pole at $\lambda = \varepsilon\sqrt{s}$, clearly F must have a simple pole at $\lambda = \varepsilon\sqrt{s}$ to cancel the pole in the first term of (A.6a). Additionally, (A.4) shows that F could have simple poles at $\lambda = -\lambda_n = -(n + \varepsilon^2 s)^{1/2}$ for $n = 0, 1, 2,...$. To factor out these poles, define

$$N(\lambda, \varepsilon^2 s) \equiv \prod_{k=1}^{\infty} [(1 + \varepsilon^2 s/k)^{1/2} + \lambda/k^{1/2}] \times e^{-2\lambda[k^{1/2} - (k-1)^{1/2}]}(1 + 1/k)^{(\lambda^2 - \varepsilon^2 s)/2} \tag{A.7}$$

The particular convergence factors chosen in (A.7) are convenient because they form a telescoping (simplifying) product. Clearly N is entire in λ—it converges uniformly in every bounded region of the complex λ plane—and the only zeros of N are simple zeros at $\lambda = -\lambda_n$ for $n = 1, 2,...$. We now write $F(\lambda, s) = E(\lambda, s)/(\lambda^2 - \varepsilon^2 s)\, N(\lambda, \varepsilon^2 s)$, so that (A.6a) becomes

$$H = \frac{e^{-\varepsilon s^{1/2} z}}{\lambda - \varepsilon\sqrt{s}} - \frac{E(\lambda, s)}{(\lambda^2 - \varepsilon^2 s)\, N(\lambda, \varepsilon^2 s)}\, U(q, z + 2\lambda)\, e^{z^2/4} \qquad \text{for} \quad z \geqslant 0 \tag{A.8}$$

Every singularity in (A.4) is now explicitly present in (A.8). Therefore E must be entire in λ. Moreover, at $\lambda = \varepsilon\sqrt{s}$ we have

$$U(q, z + 2\lambda)\, e^{z^2/4} = e^{-\varepsilon s^{1/2} z} e^{-\varepsilon^2 s}$$

So to cancel the pole at $\lambda = \varepsilon\sqrt{s}$ we must have

$$E(\lambda, s) = 2\varepsilon\sqrt{s}\, M(\varepsilon\sqrt{s})\, e^{\varepsilon^2 s} \qquad \text{at} \quad \lambda = \varepsilon\sqrt{s} \tag{A.9a}$$

where

$$M(\varepsilon\sqrt{s}) = N(\varepsilon\sqrt{s}, \varepsilon^2 s) \tag{A.9b}$$

We now show that E is constant in λ. At each z, (A.4) shows that

$$H \sim \frac{1}{\lambda}\left\{\beta_0 e^{+\varepsilon s^{1/2} z} + \sum_{n=1}^{\infty} \beta_n v_n^-(z)\right\} \qquad \text{for} \quad |\lambda| \gg 1, \quad |\arg \lambda| \neq \pi \tag{A.10}$$

On the other hand, asymptotic formulas for $N(\lambda, \varepsilon^2)$ and $U(q, z+2\lambda)$ are worked out in Appendices B and C; these formulas show that at each z

$$U(q, z+2\lambda)/(\lambda^2 - \varepsilon^2 s)\, N(\lambda, \varepsilon^2 s) \sim (2\pi)^{3/4}\, \lambda^{-7/6}\, \mathrm{Ai}(\lambda^{1/3} z) \qquad \text{for} \quad |\lambda| \gg 1, \quad |\arg \lambda| \neq \pi \tag{A.11}$$

So for (A.8) to be consistent with (A.4) (in particular, at $z=0$), we must have

$$|E(\lambda, s)/\lambda^{1/6}| \leqslant \text{const} \qquad \text{as} \quad |\lambda| \to \infty \tag{A.12}$$

Since E is entire and grows sublinearly as $\lambda \to \infty$, it must be constant in λ. Thus,

$$H = \frac{e^{-\varepsilon s^{1/2} z}}{\lambda - \varepsilon\sqrt{s}} - \frac{2\varepsilon\sqrt{s}\, e^{\varepsilon^2 s} M(\varepsilon\sqrt{s})}{(\lambda^2 - \varepsilon^2 s)\, N(\lambda, \varepsilon^2 s)}\, U(q, z+2\lambda)\, e^{z^2/4} \qquad \text{for} \quad z \geqslant 0 \tag{A.13}$$

The function $H(s, \lambda, z)$ is now completely specified, at least when $z \geqslant 0$. We now invert the transform by integrating along the Bromwich contour,

$$V(s, \bar{x}, z) = \frac{1}{2\pi i}\int_B e^{\lambda \bar{x}} H(s, \lambda, z)\, d\lambda \qquad \text{for} \quad z \geqslant 0 \tag{A.14}$$

The asymptotic formula (A.11) permits us to move the contour to $\mathrm{Re}\{\lambda\} = -\infty$, converting the integral to the sum of its residues. Since

$$U(q, z+2\lambda)\, e^{z^2/4} = v_n^-(z) \qquad \text{at} \quad \lambda = -\lambda_n \tag{A.15}$$

this yields

$$V(s, \bar{x}, z) = \beta_0 e^{-\varepsilon s^{1/2}(\bar{x} - z)} + \sum_{n=1}^{\infty} \beta_n v_n^-(z)\, e^{-\lambda_n \bar{x}} \tag{A.16}$$

with

$$\beta_0 = M(\varepsilon\sqrt{s})/M(-\varepsilon\sqrt{s}),$$
$$\beta_n = -2\varepsilon\sqrt{s}\, e^{\varepsilon^2 s} M(\varepsilon\sqrt{s})/n N_\lambda(-\lambda_n, \varepsilon^2 s) \tag{A.17}$$

Moreover, from Appendix B

$$M(-\varepsilon\sqrt{s}) = 1/M(\varepsilon\sqrt{s}),$$
$$N_\lambda(-\lambda_n, \varepsilon^2 s) = -(-1)^n\, 2\lambda_n(n-1)!/N(\lambda_n, \varepsilon^2 s) \tag{A.18}$$

Thus, the solution to the key boundary layer problem (A.1) is

$$V(s, \bar{x}, z) = \beta_0 e^{-\varepsilon s^{1/2}(\bar{x}-z)} + \sum_{n=1}^{\infty} \beta_n v_n^-(z)\, e^{-\lambda_n \bar{x}} \tag{A.19a}$$

with

$$\beta_0 = M^2(\varepsilon\sqrt{s}),$$
$$\beta_n = (-1)^n\, \varepsilon\sqrt{s}\, e^{\varepsilon^2 s} M(\varepsilon\sqrt{s})\, N(\lambda_n, \varepsilon^2 s)/\lambda_n n! \tag{A.19b}$$

A.1. Verification. Uniqueness

One can now easily verify that (A.19) is the solution of (A.1) for all z, even though (A.13) is valid only for $z \geqslant 0$. In brief, let the branch cut of $\sqrt{s}$ be along the negative real axis. Smoothness properties of (A.19) can be determined by using the asymptotic formulas in Appendices B and C to establish the uniform convergence of (A.19) and its derivatives over bounded regions. This shows that V is continuous in $\bar{x}$ and z, and analytic in s, for all s off the negative real axis and all $\bar{x} \geqslant 0$. Moreover, when $\bar{x}$ is strictly positive, then V is infinitely differentiable in $\bar{x}$ and z and the series (A.19) can be differentiated term by term. Thus, (A.19) satisfies Eq. (A.1a) since each term does. Additionally, since the transform of (A.19) is (A.13) for all $z \geqslant 0$, and since

$$\lambda H(s, \lambda, z) \sim e^{-\varepsilon s^{1/2} z} \quad \text{as} \quad |\lambda| \to \infty \quad \text{with} \quad |\arg \lambda| < \pi/2 \tag{A.20}$$

then (A.19) also satisfies (A.1b). Finally, (A.1c) is clearly satisfied, so (A.19) *is* a solution to (A.1).

It is easy to see why (A.19) must be the *unique* solution of (A.1) when $\mathrm{Re}\{s\} > 0$. To show uniqueness, we must show that $h \equiv 0$ is the only solution of

$$z h_{\bar{x}} = h_{zz} - z h_z - \varepsilon^2 s h \qquad \text{for} \quad \bar{x} > 0, \quad \text{all } z \tag{A.21a}$$

$$h(0, z) = 0 \qquad \text{for} \quad z \geqslant 0 \tag{A.21b}$$

with

$$h(\bar{x}, z) = \gamma_0 e^{-\varepsilon s^{1/2}(\bar{x}-z)} + \sum_{n=1}^{\infty} \gamma_n v_n^-(z)\, e^{-\lambda_n \bar{x}} \qquad \text{for all} \quad z \tag{A.21c}$$

But from (A.21a)

$$\langle zhh^*, 1\rangle_{\bar{x}} = -2\langle h_z h_z^*, 1\rangle - 2\varepsilon^2 \operatorname{Re}\{s\}\langle hh^*, 1\rangle \qquad \text{(A.22)}$$

where h^* is the complex conjugate of h. Integrating over all $\bar{x}$ now yields

$$\langle zh(0, z)\, h^*(0, z), 1\rangle > 0 \qquad \text{when} \quad \operatorname{Re}\{s\} > 0 \qquad \text{(A.23)}$$

unless $h(\bar{x}, z)$ is identically zero. But (A.21b) implies that the left side of (A.23) is negative or zero. Hence $h(\bar{x}, z)$ must be identically zero, and thus $V(s, \bar{x}, z)$ must be unique, for all s with $\operatorname{Re}\{s\} > 0$. The uniqueness of the solution $V(s, \bar{x}, z)$ of (A.1) (when $\operatorname{Re}\{s\} > 0$) can also be proven rigorously by applying the results in ref. 14.

We do not know if the solution $V(s, \bar{x}, z)$ of (A.1) is unique when $\operatorname{Re}\{s\} < 0$. However, V is analytic and unique for $\operatorname{Re}\{s\} > 0$, which guarantees that the inverse Laplace transform $v(t, \bar{x}, z)$ is unique. Only $v(t, \bar{x}, z)$ has direct physical meaning, so whether the solution $V(s, \bar{x}, z)$ of (A.1) is unique when $\operatorname{Re}\{s\} < 0$ is moot.

APPENDIX B. PROPERTIES OF N AND M

Here we very briefly examine

$$N(\lambda, r^2) = \prod_{k=1}^{\infty} [(1 + r^2/k)^{1/2} + \lambda/k^{1/2}]\, e^{-2\lambda[k^{1/2} - (k-1)^{1/2}]}(1 + 1/k)^{(\lambda^2 - r^2)/2} \qquad \text{(B.1a)}$$

and

$$M(r) \equiv N(r, r^2) = \prod_{k=1}^{\infty} [(1 + r^2/k)^{1/2} + r/k^{1/2}]\, e^{-2r[k^{1/2} - (k-1)^{1/2}]} \qquad \text{(B.1b)}$$

where we have set $r = \varepsilon\sqrt{s}$ for clarity.

B.1. Reflection

The infinite-product representation of the gamma function[15] yields the reflection formula

$$N(\lambda, r^2)\, N(-\lambda, r^2) = 1/\Gamma(1 + r^2 - \lambda^2) = (1/\pi) \sin \pi(\lambda^2 - r^2)\, \Gamma(\lambda^2 - r^2) \qquad \text{(B.2)}$$

Setting $\lambda_n = (n + r^2)^{1/2}$, we now obtain

$$N_\lambda(-\lambda_n, r^2) = -2(-1)^n \lambda_n (n-1)!/N(\lambda_n, r^2) \qquad \text{(B.3)}$$

as used in Appendix A. Additionally, clearly, $M(r)\, M(-r) \equiv 1$.

B.2. Integral Representations

Differentiating (B.1b) with respect to r yields

$$M'(r)/M(r) = \lim_{m \to \infty} \left\{ \sum_{k=1}^{m} (k+r^2)^{-1/2} - 2\sqrt{m} \right\} = \zeta(\tfrac{1}{2}, 1+r^2) \quad \text{(B.4)}$$

where ζ is the generalized Riemann zeta function.[15] So M can be represented as

$$M(r) = \exp\left\{ \int_0^r \zeta(\tfrac{1}{2}, 1+q^2)\, dq \right\} = \exp\left\{ r \int_0^1 \zeta(\tfrac{1}{2}, 1+r^2q^2)\, dq \right\} \quad \text{(B.5a)}$$

Similarly,

$$N(\lambda_n, r^2) = N(\sqrt{n}, 0) \exp\left\{ \tfrac{1}{2}r^2 \int_0^1 \zeta(\tfrac{1}{2}, 1+qr^2)/(n+qr^2)^{1/2}\, dq \right\} \quad \text{(B.5b)}$$

where $\lambda_n = (n+r^2)^{1/2}$ as before. In particular,

$$\zeta(\tfrac{1}{2}, 1+q) = \zeta(\tfrac{1}{2}) - \tfrac{1}{2}q\zeta(\tfrac{3}{2}) + \tfrac{3}{8}q^2\zeta(\tfrac{5}{2}) - \cdots \quad \text{(B.6)}$$

so

$$M(r) = \exp\{\zeta(\tfrac{1}{2})r - \tfrac{1}{6}\zeta(\tfrac{3}{2})\, r^3 + \tfrac{3}{40}\zeta(\tfrac{5}{2})\, r^5 - \cdots\} \quad \text{(B.7a)}$$

and

$$N(\lambda_n, r^2) = N(\sqrt{n}, 0) \exp\{\zeta(\tfrac{1}{2})\, r^2/2\sqrt{n} + \cdots\} \quad \text{(B.7b)}$$

for $|r| \ll 1$, as used in Sections 4 and 5.

B.3. Large-λ Asymptotics

Consider the partial product of (B.1a),

$$\log N_n = -2\lambda\sqrt{n} + \tfrac{1}{2}(\lambda^2 - r^2)\log(n+1) + n\log\lambda - \tfrac{1}{2}\log n! + \sum_{k=1}^{n} f(k) \quad \text{(B.8a)}$$

with

$$f(k) = \log[1 + (k+r^2)^{1/2}/\lambda] \quad \text{(B.8b)}$$

To analyze (B.8) for large λ, we use the intermediate limit

$$\sum_{k=1}^{n} f(k) = \sum_{k=1}^{m-1} f(k) + \sum_{k=m}^{n} f(k) \quad \text{(B.9)}$$

where $m \gg 1$ and $\sqrt{m}/\lambda \ll 1$. Then for n large enough,

$$\sum_{k=m}^{n} f(k) = \int_m^n f(k)\, dk + \tfrac{1}{2} f(n) + O(\sqrt{m}/\lambda) + O(1/n) \tag{B.10}$$

provided that $|\arg \lambda| < \pi$. Substituting (B.9) and (B.10) into (B.8) and letting $n \to \infty$ now yields

$$\log N(\lambda, r^2) = -\tfrac{1}{4} \log 2\pi + (\lambda^2 - r^2 - \tfrac{1}{2}) \log \lambda - \tfrac{1}{2}\lambda^2 + D_m(\lambda) \tag{B.11a}$$

where

$$D_m(\lambda) = \sum_{k=1}^{m-1} f(k) + (\lambda^2 - r^2 - m) f(m) - \lambda(m + r^2)^{1/2}$$
$$+ \tfrac{1}{2}(m + r^2) + O(m^{1/2}/\lambda) \tag{B.11b}$$

Since $m \gg 1$ and $m^{1/2}/\lambda \ll 1$, we have $D_m(\lambda) \ll 1$. So

$$N(\lambda, r^2) \sim (2\pi)^{-1/4} \lambda^{-1/2} \exp\{(\lambda^2 - r^2) \log \lambda - \tfrac{1}{2}\lambda^2\} \tag{B.12}$$

for $|\lambda| \gg 1$ with $|\arg \lambda| < \pi$.

APPENDIX C. ASYMPTOTICS OF *U*

To analyze U for large λ, recall that[13]

$$U(q, z + 2\lambda) = \Gamma(1 + \lambda^2 - r^2)\, e^{-(z+2\lambda)^2/4} \frac{1}{2\pi i} \int_H e^{zs + 2\lambda s - s^2/2} s^{r^2 - \lambda^2 - 1}\, ds \tag{C.1}$$

where H is the Hankel contour and where we have set $r = \varepsilon\sqrt{s}$ for brevity. We evaluate (C.1) by the method of steepest descent. Let

$$s = \lambda + \lambda^{1/3} t, \qquad y = \lambda^{1/3} z \tag{C.2}$$

Then for $|\lambda| \gg 1$ with $|\arg \lambda| < \pi$,

$$U(q, z + 2\lambda) = \lambda^{4/3} \Gamma(\lambda^2 - r^2)\, e^{-(\lambda^2 - r^2)\log \lambda} e^{+\lambda^2/2} S(\lambda, y) \tag{C.3a}$$

with

$$S(\lambda, y) = \frac{1}{2\pi i} \int_{H'} \exp(yt - \tfrac{1}{3}t^3)[1 + O(\lambda^{-2/3})]\, dt \tag{C.3b}$$

Deforming the contour onto the imaginary axis now yields Airy's function[13]

$$S(\lambda, y) \sim \mathrm{Ai}(y) + O(\lambda^{-2/3}) \tag{C.4}$$

So when $y = \lambda^{1/3} z = O(1)$,

$$U(q, z + 2\lambda) \sim (2\pi)^{1/2}\, \lambda^{1/3} e^{(\lambda^2 - r^2)\log \lambda} e^{-\lambda^2/2}\, \mathrm{Ai}(\lambda^{1/3} z) \tag{C.5}$$

for $|\lambda| \gg 1$ with $|\arg \lambda| < \pi$. However, $U(q, z + 2\lambda)$ solves

$$U_{zz} = (\lambda z + z^2/4 + r^2 - \tfrac{1}{2})\, U \tag{C.6}$$

Now (C.6) has no distinguished limits between $\lambda^{1/3} z = O(1)$ and $z = O(1)$. Therefore (C.5) is also valid as $|\lambda| \to \infty$ with $|\arg \lambda| < \pi$ and with z fixed.

Finally, similar arguments show that for any fixed k

$$\partial_z^k U(q, z + 2\lambda) \sim (2\pi)^{1/2}\, \lambda^{1/3} e^{(\lambda^2 - r^2)\log \lambda} e^{-\lambda^2/2} \partial_z^k\, \mathrm{Ai}(\lambda^{1/3} z) \tag{C.7}$$

for $|\lambda| \gg 1$ with $|\arg \lambda| < \pi$.

ACKNOWLEDGMENTS

We acknowledge useful discussions with Profs. J. N. Franklin and G. B. Whitham. This research was supported by the U.S. DOE under the Applied Mathematical Sciences Research Program KC-07-01-01-0, by grant W-7405-ENG-36, and by the Office of Scientific Computing. Also supported by AFOSR FF-4962086-C0130 under the University Research Initiative.

REFERENCES

1. L. Arnold, *Stochastic Differential Equations* (Wiley, New York, 1974).
2. N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
3. G. Blankenship and G. C. Papanicolaou, *SIAM J. Appl. Math.* **34**:437 (1978).
4. W. Horsthemke and R. Lefever, *Noise Induced Transitions* (Springer, New York, 1984).
5. J. L. Doob, *Ann. Math.* **43**:351 (1942).
6. G. E. Uhlenbeck and L. S. Ornstein, *Phys. Rev.* **36**:823 (1930).
7. M. C. Wang and G. E. Uhlenbeck, *Rev. Mod. Phys.* **17**:323 (1945).
8. E. Larsen and J. B. Keller, *J. Math. Phys.* **15**:75 (1974).
9. M. A. Burschka and U. M. Titulaer, *J. Stat. Phys.* **25**:569 (1981).
10. S. Harris, *J. Chem. Phys.* **75**:3103 (1981).
11. T. W. Marshall and E. J. Watson, *J. Phys. A* **18**:3531 (1985).
12. P. S. Hagan, C. R. Doering, and C. D. Levermore, Mean exit times driven by weakly colored noise, *SIAM J. Appl. Math.*, to appear.

13. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D.C., 1964).
14. W. Greenberg, C. V. M. van der Mee, and V. Protopopescu, *Boundary Value Problems in Abstract Kinetic Theory* (Birkhauser, Basel, 1987).
15. G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a Complex Variable* (McGraw-Hill, New York, 1966).
16. J. M. Sancho, M. San Miguel, S. L. Katz, and J. D. Gunton, *Phys. Rev. A* **26**:1589 (1982).
17. P. Hänggi, T. J. Mroczkowski, F. Moss, and P. V. E. McClintock, *Phys. Rev. A* **33**:695 (1986).
18. R. F. Fox, *Phys. Rev. A* **33**:467 (1986).
19. J. Masoliver, B. J. West, and K. Lindenberg, *Phys. Rev. A* **35**:3086 (1987).